Midterm 2 Solution

Problem 1

Let s_i equal the outcome of the student's quiz. The sample space is then composed of all the possible grades that she can receive.

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \tag{1}$$

Since each of the 11 possible outcomes is equally likely, the probability of receiving a grade of i, for each $i = 0, 1, \ldots, 10$ is $P[s_i] = 1/11$. The probability that the student gets an A is the probability that she gets a score of 9 or higher. That is

$$P[\text{Grade of A}] = P[9] + P[10] = 1/11 + 1/11 = 2/11. \tag{2}$$

The probability of failing requires the student to get a grade less than 4.

$$P[\text{Failing}] = P[3] + P[2] + P[1] + P[0] = 1/11 + 1/11 + 1/11 + 1/11 = 4/11. \tag{3}$$

Problem 2

In this experiment, there are four outcomes with probabilities

$$P[\{vv\}] = (0.8)^2 = 0.64 \qquad P[\{vd\}] = (0.8)(0.2) = 0.16$$
$$P[\{dv\}] = (0.2)(0.8) = 0.16 \qquad P[\{dd\}] = (0.2)^2 = 0.04$$

When checking the independence of any two events A and B, it's wise to avoid intuition and simply check whether $P[AB] = P[A]P[B]$. Using the probabilities of the outcomes, we now can test for the independence of events.

(1) First, we calculate the probability of the joint event:

$$P[N_V = 2, N_V \geq 1] = P[N_V = 2] = P[\{vv\}] = 0.64 \tag{1}$$

Next, we observe that

$$P[N_V \geq 1] = P[\{vd, dv, vv\}] = 0.96 \tag{2}$$

Finally, we make the comparison

$$P[N_V = 2]\, P[N_V \geq 1] = (0.64)(0.96) \neq P[N_V = 2, N_V \geq 1] \tag{3}$$

which shows the two events are dependent.

(2) The probability of the joint event is

$$P[N_V \geq 1, C_1 = v] = P[\{vd, vv\}] = 0.80 \tag{4}$$

From part (a), $P[N_V \geq 1] = 0.96$. Further, $P[C_1 = v] = 0.8$ so that

$$P[N_V \geq 1]\, P[C_1 = v] = (0.96)(0.8) = 0.768 \neq P[N_V \geq 1, C_1 = v] \tag{5}$$

Hence, the events are dependent.

(3) The problem statement that the calls were independent implies that the events the second call is a voice call, $\{C_2 = v\}$, and the first call is a data call, $\{C_1 = d\}$ are independent events. Just to be sure, we can do the calculations to check:

$$P[C_1 = d, C_2 = v] = P[\{dv\}] = 0.16 \tag{6}$$

Since $P[C_1 = d]P[C_2 = v] = (0.2)(0.8) = 0.16$, we confirm that the events are independent. Note that this shouldn't be surprising since we used the information that the calls were independent in the problem statement to determine the probabilities of the outcomes.

(4) The probability of the joint event is

$$P[C_2 = v, N_V \text{ is even}] = P[\{vv\}] = 0.64 \tag{7}$$

Also, each event has probability

$$P[C_2 = v] = P[\{dv, vv\}] = 0.8, \quad P[N_V \text{ is even}] = P[\{dd, vv\}] = 0.68 \tag{8}$$

Thus, $P[C_2 = v]P[N_V \text{ is even}] = (0.8)(0.68) = 0.544$. Since $P[C_2 = v, N_V \text{ is even}] \neq 0.544$, the events are dependent.

<u>Problem 3</u>

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is $4^3 = 64$. If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of $4 \cdot 3 \cdot 2 = 24$ possible codes.

<u>Problem 4</u>

(a) We must choose c to make the PMF of V sum to one.

$$\sum_{v=1}^{4} P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1 \tag{1}$$

Hence $c = 1/30$.

(b) Let $U = \{u^2 | u = 1, 2, \ldots\}$ so that

$$P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30} \tag{2}$$

(c) The probability that V is even is

$$P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3} \tag{3}$$

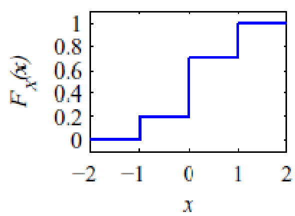
(d) The probability that $V > 2$ is

$$P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6} \tag{4}$$

Problem 5

Problem 2.4.2 Solution

(a) The given CDF is shown in the diagram below.



$$F_X(x) = \begin{cases} 0 & x < -1 \\ 0.2 & -1 \le x < 0 \\ 0.7 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 6

From the solution to Problem 2.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The PMF of $V = |X|$ satisfies

$$P_V(v) = P[|X| = v] = P_X(v) + P_X(-v) \tag{2}$$

In particular,

$$P_V(0) = P_X(0) = 0.5 \qquad P_V(1) = P_X(-1) + P_X(1) = 0.5 \tag{3}$$

The complete expression for the PMF of V is

$$P_V(v) = \begin{cases} 0.5 & v = 0 \\ 0.5 & v = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) From the PMF, we can construct the staircase CDF of V.

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 0.5 & 0 \le v < 1 \\ 1 & v \ge 1 \end{cases} \tag{5}$$

(c) From the PMF $P_V(v)$, the expected value of V is

$$E[V] = \sum_v P_V(v) = 0(1/2) + 1(1/2) = 1/2 \tag{6}$$

You can also compute $E[V]$ directly by using Theorem 2.10.

Problem 7

The PMF $P_N(n)$ allows to calculate each of the desired quantities.

(1) The expected value of N is

$$E[N] = \sum_{n=0}^{2} n P_N(n) = 0(0.1) + 1(0.4) + 2(0.5) = 1.4 \tag{1}$$

(2) The second moment of N is

$$E\left[N^2\right] = \sum_{n=0}^{2} n^2 P_N(n) = 0^2(0.1) + 1^2(0.4) + 2^2(0.5) = 2.4 \tag{2}$$

(3) The variance of N is

$$\text{Var}[N] = E\left[N^2\right] - (E[N])^2 = 2.4 - (1.4)^2 = 0.44 \tag{3}$$

(4) The standard deviation is $\sigma_N - \sqrt{\text{Var}[N]} - \sqrt{0.44} - 0.663$.

Problem 8

Each of these probabilities can be read off the CDF $F_Y(y)$. However, we must keep in mind that when $F_Y(y)$ has a discontinuity at y_0, $F_Y(y)$ takes the upper value $F_Y(y_0^+)$.

(1) $P[Y < 1] = F_Y(1^-) = 0$

(2) $P[Y \leq 1] = F_Y(1) = 0.6$

(3) $P[Y > 2] = 1 - P[Y \leq 2] = 1 - F_Y(2) = 1 - 0.8 = 0.2$

(4) $P[Y \geq 2] = 1 - P[Y < 2] = 1 - F_Y(2^-) = 1 - 0.6 = 0.4$

(5) $P[Y = 1] = P[Y \leq 1] - P[Y < 1] = F_Y(1^+) - F_Y(1^-) = 0.6$

(6) $P[Y = 3] = P[Y \leq 3] - P[Y < 3] = F_Y(3^+) - F_Y(3^-) = 0.8 - 0.8 = 0$

Extra credit Problem

The $P[-\,|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P\left[-|H\right] = P\left[+|H^c\right] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P\left[H|+\right] = \frac{P\left[+,H\right]}{P\left[+\right]} = \frac{P\left[+,H\right]}{P\left[+,H\right] + P\left[+,H^c\right]}. \tag{2}$$

We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} P\left[H|+\right] &= \frac{P\left[+|H\right]P\left[H\right]}{P\left[+|H\right]P\left[H\right] + P\left[+|H^c\right]P\left[H^c\right]} && (3)\\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} && (4)\\ &= 0.0194. && (5) \end{aligned}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.